1.3 – Matrices and Matrix Operations

Definitions: A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix. The **size** of a matrix that has m rows and n columns is $m \times n$ (read "m by n").

#6 Use the following matrices to compute the indicated expression if it is defined.

$$\underset{3\times 2}{A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}},\ B = \overset{2\times 2}{\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}},\ C = \underset{2\times 3}{\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}},\ D = \underset{3\times 3}{\begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}},\ E = \underset{3\times 3}{\begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}}$$

a. $\left(2D^T - E\right) A$
b. $(4B)C + 2B$
c. $(-AC)^T + 5D^T$
d. $\left(BA^T - 2C\right)^T$
e. $B^T(CC^T - A^TA)$
f. $D^TE^T - (ED)^T$

Definition: If A is any $m \times n$ matrix, then the **transpose of A**, denoted by A^T, is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A; that is, the first column of A^T is the first row of A, the second column of A^T is the second row of A, and so forth.

$$D^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$$

$2D^T$ is scalar multiplication

$$2D^T = \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix}$$

Addition/subtraction are performed entry by entry

$$2D^T - E = \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

$$2D^T - E = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

$3 \times \boxed{3 \quad 3} \times 2$

$$(2D^T - E)A = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12+3+3 & 0-6+3 \\ 33+1+2 & 0-2+2 \\ 0-1+5 & 0+2+5 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$$

3×2

$$A(2D^T - E) = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

is not defined

b) $(4B)C + 2B$

$$4\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix} + 2\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$$

$2 \times \boxed{2 \quad 2} \times 3$ ✓ $\qquad\qquad 2 \times 2$

Addition/subtraction is only defined for matrices that are the same size. This operation is not defined.

We computed the product

$$\underset{A}{\begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}}\underset{B}{\begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} -12+3+3 & 0-6+3 \\ 33+1+2 & 0-2+2 \\ 0-1+5 & 0+2+5 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$$

using the <u>row-column rule</u>.

Relabeling $A = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$

<u>row vectors</u> of A are (or row matrices)

$$\vec{r}_1 = [-4 \quad -3 \quad 3]$$
$$\vec{r}_2 = [11 \quad -1 \quad 2]$$
$$\vec{r}_3 = [0 \quad 1 \quad 5]$$

Column vectors of B are $\vec{c}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$

(or column vectors)

AB can also be found using the column method

$$AB = \left[A\vec{c}_1 \;\middle|\; A\vec{c}_2 \right]$$ this is a partitioned matrix.

$$A\vec{c}_1 = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 36 \\ 4 \end{bmatrix}$$ – the 1^{st} column of AB

$A\vec{c}_2$ will likewise give the 2^{nd} column of AB.

The row method: $$AB = \begin{bmatrix} \vec{r}_1 B \\ \vec{r}_2 B \\ \vec{r}_3 B \end{bmatrix}$$

#17 Use the column-row expansion of AB to express this product as a sum of matrix products.

$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$

Definition : The **column-row expansion** of AB is $AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \ldots + \mathbf{c}_r\mathbf{r}_r$, where $\mathbf{c}_i$ are column vectors of A and $\mathbf{r}_i$ are row vectors of B.

$$AB = \begin{bmatrix} 4 \\ 2 \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \end{bmatrix}$$

$2 \times 1 \quad 1 \times 3$

$$= \begin{bmatrix} 0 & 4 & 8 \\ 0 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & -9 & -3 \\ 2 & -3 & -1 \end{bmatrix}$$

sum of matrix products

$$= \begin{bmatrix} 6 & -5 & 5 \\ 2 & -1 & 3 \end{bmatrix}$$

The linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned}$$

Can be expressed using matrix multiplication.

$m \times n \quad n \times 1 \qquad m \times 1$

Definition: When a linear system is written using a matrix product

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

or $A\mathbf{x} = \mathbf{b}$, A is called the **coefficient matrix** of the system.

#13 a. Express the matrix equation as a system of linear equations.

$$\underbrace{\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}}_{\vec{b}} \qquad \begin{bmatrix} 5x_1 + 6x_2 - 7x_3 \\ -x_1 - 2x_2 + 3x_3 \\ 4x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{aligned} 5x_1 + 6x_2 - 7x_3 &= 2 \\ -x_1 - 2x_2 + 3x_3 &= 0 \\ 4x_2 - x_3 &= 3 \end{aligned}$$

Definition: Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal.

Definition: If $A_1, A_2, \ldots, A_r$ are matrices of the same size, and if $c_1, c_2, \ldots, c_r$ are scalars, then an expression of the form $c_1A_1 + c_2A_2 + \ldots + c_rA_r$ is called a **linear combination** of $A_1, A_2, \ldots, A_r$ with **coefficients** $c_1, c_2, \ldots, c_r$.

Above, we have $A = \begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix}$,

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$

Note that on the left-hand side

$$\begin{bmatrix} 5x_1+6x_2-7x_3 \\ -x_1-2x_2+3x_3 \\ 4x_2-x_3 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ -x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 6x_2 \\ -2x_2 \\ 4x_2 \end{bmatrix} + \begin{bmatrix} -7x_3 \\ 3x_3 \\ -x_3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} -7 \\ 3 \\ -1 \end{bmatrix} x_3$$

linear combination of the columns of A

#22 Example 6 of section 1.2 presents the linear system

$$\begin{array}{rrrrrrl} x_1 & +3x_2 & -2x_3 & & +2x_5 & & = 0 \\ 2x_1 & +6x_2 & -5x_3 & -2x_4 & +4x_5 & -3x_6 & = 0 \\ & & 5x_3 & +10x_4 & & +15x_6 & = 0 \\ 2x_1 & +6x_2 & & +8x_4 & +4x_5 & +18x_6 & = 0 \end{array}$$

and the reduced row echelon form of its associated augmented matrix as

$$\left[\begin{array}{cccccc|c} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -3x_2 - 4x_4 - 2x_5$

$x_3 = -2x_4$

$x_6 = 0$

Express the solution as a linear combination of column vectors that contain only numerical entries.

Let $r = x_2$, $s = x_4$, $t = x_5$.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r-4s-2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} t$$

Theorem 1.3.1 If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.

Definition: The **main diagonal** of a square matrix contains entries from the upper left to lower right corners.

Definition: If A is a square matrix, then the **trace of A**, denoted by $tr\,(A)$, is defined to be the sum of the entries on the main diagonal of A. The trace of A is undefined if A is not a square matrix.

A^T - transpose

$tr(A)$ - trace